Extremal problems— Examples

Proposition 1. If G is an n-vertex graph with at most $n-2$ edges then G is disconnected.

Proof. By induction on $e(G)$ prove that every graph G has at least $n(G) - e(G)$ components.

A **Question** you always have to ask:
Can we improve on this proposition?

Answer. NO! The same statement is FALSE with $n-1$ in the place of $n-2$. Proposition 1 is *best possible*, because P_n has $n-1$ edges and is not disconnected.

Proposition 2. If G is an n-vertex graph with at least n edges then G contains a cycle.

Proof: Induction. + **Lemma.** If every vertex of a multigraph G has degree at least 2, then G contains a cycle.

Remark. Proposition 2 is also *best possible*, (e.g. P_n).

Proposition 1. + Remark: The minimum value of $e(G)$ over connected graphs is $n-1$.

Proposition 2. + Remark: The maximum value of $e(G)$ over acyclic (i.e. cycle-free) graphs is $n-1$.

Extremal problems: Connectivity

Vague description: An extremal problem asks for the maximum or minimum value of a parameter over a class of graphs (or other discrete structures).

Question: What is the smallest possible minimum degree that GUARANTEES connectivity?

Conjecture by Construction $K_{\lfloor n/2 \rfloor} + K_{\lceil n/2 \rceil}$ is disconnected and has "large" minimum degree ($\lfloor n/2 \rfloor - 1$). Is this best possible?

> **Def.** Graph $G + H$ is the disjoint union (or sum) of graphs G and H.

Proposition. G is an n-vertex graph with $\delta(G) \geq \lfloor n/2 \rfloor$, then G is connected.

Prop. + Construction: The maximum value of $\delta(G)$ over disconnected graphs is $\lfloor \frac{n}{2} \rfloor - 1$.

Extremal problems: Hamiltonicity

Hamilton cycle: a cycle going through all vertices of graph G. G is Hamiltonian: G contains a Hamilton cycle

Examples: Dodecaeder graph, Petersen graph

Remark. Hamiltonicity is difficult to decide; special case of Travelling Salesman Problem (TSP)

Question. How much larger min-degree *forces* Hamiltonicity than connectivity? Not much more!

Dirac's Theorem G is an n-vertex graph with $\delta(G) \geq \left\lfloor \frac{n+1}{2} \right\rfloor$, then G is Hamiltonian.

Proof. Take longest path and prove that there is a cycle spanning all its vertices. Conclude this must be a Hamilton cycle otherwise the path can be lengthened.

Remark. The above proposition is *best possible*, as $K_1 \vee (K_{\lfloor (n-1)/2 \rfloor} + K_{\lceil (n-1)/2 \rceil})$ has minimum degree $\lfloor (n-1)/2 \rfloor$ and no Hamilton cycle.

Graph $G \vee H$ is the join of graphs G and H: take $G + H$ and add all edges between $V(G)$ and $V(H)$.

Extremal Problems

graph property	graph parameter	type of extremum	value of extremum
connected	$e(G)$	min	$n-1$
acyclic	$e(G)$	max	$n-1$
disconnected	$\delta(G)$	max	$\left\lfloor \frac{n}{2} \right\rfloor - 1$
non-Hamiltonian	$\delta(G)$	max	$\left\lfloor \frac{n-1}{2} \right\rfloor$
K_3-free	$e(G)$	max	$\left\lfloor \frac{n^2}{4} \right\rfloor$

Triangle-free subgraphs

Theorem. (Mantel, 1907) The maximum number of edges in an n-vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$.

Proof.

(i) *There is* a triangle-free graph with $\lfloor \frac{n^2}{4} \rfloor$ edges.

(ii) If G is a triangle-free graph, then $e(G) \leq \lfloor \frac{n^2}{4} \rfloor$.

Proof of (ii): estimate edges of a K_3-free graph by summing up degrees of the vertices in the complement the neighborhood of a maximum degree vertex